SERIES — Answers

1 **a** $= 1 + (\tfrac{1}{2})(-4x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-4x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(-4x)^3 + \ldots$

$= 1 - 2x - 2x^2 - 4x^3 + \ldots,\ |-4x| < 1 \quad \therefore$ valid for $|x| < \tfrac{1}{4}$

b when $x = 0.01$, $(1 - 4x)^{\frac{1}{2}} \approx 1 - 2(0.01) - 2(0.01)^2 - 4(0.01)^3$

$= 1 - 0.02 - 0.0002 - 0.000\,004$

$= 0.979\,796$

$(1 - 0.04)^{\frac{1}{2}} = \sqrt{0.96} = \sqrt{\frac{16\times6}{100}} = \tfrac{2}{5}\sqrt{6}$

$\therefore\ \sqrt{6} \approx \tfrac{5}{2} \times 0.979\,796 = 2.44949$ (6sf)

2 **a** $\frac{4}{1+2x-3x^2} \equiv \frac{4}{(1+3x)(1-x)} \equiv \frac{A}{1+3x} + \frac{B}{1-x}$

$4 \equiv A(1-x) + B(1+3x)$

$x = -\tfrac{1}{3} \quad \Rightarrow \quad 4 = \tfrac{4}{3}A \quad \Rightarrow \quad A = 3$

$x = 1 \quad \Rightarrow \quad 4 = 4B \quad \Rightarrow \quad B = 1$

$\therefore\ f(x) = \frac{3}{1+3x} + \frac{1}{1-x}$

b $\frac{3}{1+3x} = 3(1+3x)^{-1} = 3[1 + (-1)(3x) + \frac{(-1)(-2)}{2}(3x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(3x)^3 + \ldots]$

$= 3 - 9x + 27x^2 - 81x^3 + \ldots,\ |3x| < 1 \therefore$ valid for $|x| < \tfrac{1}{3}$

$\frac{1}{1-x} = (1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(-x)^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots,\ |-x| < 1 \therefore$ valid for $|x| < 1$

$\therefore\ f(x) = (3 - 9x + 27x^2 - 81x^3 + \ldots) + (1 + x + x^2 + x^3 + \ldots)$

$= 4 - 8x + 28x^2 - 80x^3 + \ldots,$ valid for $|x| < \tfrac{1}{3}$

3 **a** $= 2^{-2}(1 - \tfrac{1}{2}x)^{-2} = \tfrac{1}{4}(1 - \tfrac{1}{2}x)^{-2}$

$= \tfrac{1}{4}[1 + (-2)(-\tfrac{1}{2}x) + \frac{(-2)(-3)}{2}(-\tfrac{1}{2}x)^2 + \frac{(-2)(-3)(-4)}{3\times2}(-\tfrac{1}{2}x)^3 + \ldots]$

$= \tfrac{1}{4} + \tfrac{1}{4}x + \tfrac{3}{16}x^2 + \tfrac{1}{8}x^3 + \ldots$

b $\frac{3-x}{(2-x)^2} = (3-x)(2-x)^{-2} = (3-x)(\tfrac{1}{4} + \tfrac{1}{4}x + \tfrac{3}{16}x^2 + \tfrac{1}{8}x^3 + \ldots)$

$\therefore$ coefficient of $x^3 = (3 \times \tfrac{1}{8}) + (-1 \times \tfrac{3}{16}) = \tfrac{3}{16}$

4 **a** $f(\tfrac{1}{10}) = \frac{4}{\sqrt{1+\frac{1}{15}}} = \frac{4}{\sqrt{\frac{16}{15}}} = \frac{4}{\frac{4}{\sqrt{15}}} = 4 \times \frac{\sqrt{15}}{4} = \sqrt{15}$

b $= 4(1 + \tfrac{2}{3}x)^{-\frac{1}{2}} = 4[1 + (-\tfrac{1}{2})(\tfrac{2}{3}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\tfrac{2}{3}x)^2 + \ldots]$

$= 4 - \tfrac{4}{3}x + \tfrac{2}{3}x^2 + \ldots$

c $\sqrt{15} = f(\tfrac{1}{10}) \approx 4 - \tfrac{4}{3}\times\tfrac{1}{10} + \tfrac{2}{3}\times(\tfrac{1}{10})^2 + \ldots$

$= 4 - \tfrac{2}{15} + \tfrac{1}{150} = 3\tfrac{131}{150}$

d $\sqrt{15} = 3.872\,98\ldots$

$3\tfrac{131}{150} = 3.873\,33\ldots$

$3\tfrac{55}{63} = 3.873\,01\ldots$

$\therefore\ \sqrt{15} < 3\tfrac{55}{63} < 3\tfrac{131}{150}$, so $3\tfrac{55}{63}$ is a more accurate approximation

5 **a** $= 1 + (\frac{1}{3})(-x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(-x)^2 + \ldots$

$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 + \ldots$

b when $x = 10^{-3}$, $(1 - x)^{\frac{1}{3}} \approx 1 - \frac{1}{3}(10^{-3}) - \frac{1}{9}(10^{-3})^2$

$= 0.999\,666\,555\,6$

$(1 - 10^{-3})^{\frac{1}{3}} = \sqrt[3]{0.999} = \sqrt[3]{\frac{27\times37}{1000}} = \frac{3}{10}\sqrt[3]{37}$

$\therefore \sqrt[3]{37} \approx \frac{10}{3} \times 0.999\,666\,555\,6 = 3.332\,221\,85$ (9sf)

6 **a** $p = \frac{(\frac{3}{5})(-\frac{2}{5})}{2}(5)^2 = -3$

$q = \frac{(\frac{3}{5})(-\frac{2}{5})(-\frac{7}{5})}{3\times2}(5)^3 = 7$

b let $x = 0.02$

$(1.1)^{\frac{3}{5}} \approx 1 + 3(0.02) - 3(0.02)^2 + 7(0.02)^3$

$= 1 + 0.06 - 0.0012 + 0.000\,056$

$= 1.058\,856$

c $(1.1)^{\frac{3}{5}} = 1.058\,852\,853\ldots$

% error $= \frac{1.058856 - 1.058852853}{1.058852853} \times 100\% = 0.000\,297$ % (3sf)

7 **a** $8 - 6x^2 \equiv A(2 + x)^2 + B(1 + x)(2 + x) + C(1 + x)$

$x = -1 \quad\Rightarrow\quad A = 2$

$x = -2 \quad\Rightarrow\quad -16 = -C \quad\Rightarrow\quad C = 16$

coeffs of $x^2 \quad\Rightarrow\quad -6 = A + B \quad\Rightarrow\quad B = -8$

b $\frac{8-6x^2}{(1+x)(2+x)^2} \equiv \frac{2}{1+x} - \frac{8}{2+x} + \frac{16}{(2+x)^2}$

$\frac{2}{1+x} = 2(1 + x)^{-1} = 2[1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3\times2}x^3 + \ldots]$

$= 2 - 2x + 2x^2 - 2x^3 + \ldots$

$\frac{8}{2+x} = 8(2 + x)^{-1} = 8 \times 2^{-1}(1 + \frac{1}{2}x)^{-1} = 4(1 + \frac{1}{2}x)^{-1}$

$= 4[1 + (-1)(\frac{1}{2}x) + \frac{(-1)(-2)}{2}(\frac{1}{2}x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(\frac{1}{2}x)^3 + \ldots]$

$= 4 - 2x + x^2 - \frac{1}{2}x^3 + \ldots$

$\frac{16}{(2+x)^2} = 16(2 + x)^{-2} = 16 \times 2^{-2}(1 + \frac{1}{2}x)^{-2} = 4(1 + \frac{1}{2}x)^{-2}$

$= 4[1 + (-2)(\frac{1}{2}x) + \frac{(-2)(-3)}{2}(\frac{1}{2}x)^2 + \frac{(-2)(-3)(-4)}{3\times2}(\frac{1}{2}x)^3 + \ldots]$

$= 4 - 4x + 3x^2 - 2x^3 + \ldots$

$\therefore \frac{8-6x^2}{(1+x)(2+x)^2} = (2 - 2x + 2x^2 - 2x^3 + \ldots) - (4 - 2x + x^2 - \frac{1}{2}x^3 + \ldots) + (4 - 4x + 3x^2 - 2x^3 + \ldots)$

$= 2 - 4x + 4x^2 - \frac{7}{2}x^3 + \ldots$

8 **a** $= 1 + (\tfrac{1}{2})(-2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-2x)^2 + \ldots$

$= 1 - x - \tfrac{1}{2}x^2 + \ldots$

b when $x = 0.0008$, $(1 - 2x)^{\frac{1}{2}} \approx 1 - 0.0008 - \tfrac{1}{2}(0.0008)^2$

$= 1 - 0.0008 - 0.000\,000\,32$

$= 0.999\,199\,68$

$(1 - 0.0016)^{\frac{1}{2}} = \sqrt{0.9984} = \sqrt{\frac{256\times 39}{10000}} = \frac{4}{25}\sqrt{39}$

$\therefore \sqrt{39} \approx \frac{25}{4} \times 0.999\,199\,68 = 6.244\,998$ (7sf)

9 **a** $= 1 + (\tfrac{1}{3})(8x) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}(8x)^2 + \ldots$

$= 1 + \tfrac{8}{3}x - \tfrac{64}{9}x^2 + \ldots$

b $k = \sqrt[3]{\frac{5}{1.08}} = \sqrt[3]{\frac{500}{108}} = \sqrt[3]{\frac{125}{27}} = \frac{5}{3}$

c let $x = 0.01$, $\sqrt[3]{1.08} = 1 + \tfrac{8}{3}(0.01) - \tfrac{64}{9}(0.01)^2$

$= 1.025\,955\,556$

$\therefore \sqrt[3]{5} = \frac{5}{3} \times 1.025\,955\,556 = 1.710$ (4sf)

10 **a** $\mathrm{f}(x) \equiv \frac{6x}{(x-1)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x-3}$

$6x \equiv A(x-3) + B(x-1)$

$x = 1 \Rightarrow 6 = -2A \Rightarrow A = -3$

$x = 3 \Rightarrow 18 = 2B \Rightarrow B = 9$

$\mathrm{f}(x) \equiv \frac{9}{x-3} - \frac{3}{x-1}$

b $\mathrm{f}(x) = \frac{3}{1-x} - \frac{9}{3-x}$

$\frac{3}{1-x} = 3(1-x)^{-1} = 3[1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-x)^3 + \ldots]$

$= 3 + 3x + 3x^2 + 3x^3 + \ldots$

$\frac{9}{3-x} = 9(3-x)^{-1} = 9 \times 3^{-1}(1 - \tfrac{1}{3}x)^{-1} = 3(1 - \tfrac{1}{3}x)^{-1}$

$= 3[1 + (-1)(-\tfrac{1}{3}x) + \frac{(-1)(-2)}{2}(-\tfrac{1}{3}x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-\tfrac{1}{3}x)^3 + \ldots]$

$= 3 + x + \tfrac{1}{3}x^2 + \tfrac{1}{9}x^3 + \ldots$

$\therefore \mathrm{f}(x) = (3 + 3x + 3x^2 + 3x^3 + \ldots) - (3 + x + \tfrac{1}{3}x^2 + \tfrac{1}{9}x^3 + \ldots)$

$= 2x + \tfrac{8}{3}x^2 + \tfrac{26}{9}x^3 + \ldots$

$\therefore$ for small x, $\mathrm{f}(x) \approx 2x + \tfrac{8}{3}x^2 + \tfrac{26}{9}x^3$

11 **a** $= 4^{\frac{1}{2}}(1 + \frac{1}{4}x)^{\frac{1}{2}} = 2(1 + \frac{1}{4}x)^{\frac{1}{2}} = 2[1 + (\frac{1}{2})(\frac{1}{4}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\frac{1}{4}x)^2 + \ldots]$

$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \ldots,\ |\frac{1}{4}x| < 1 \quad \therefore$ valid for $|x| < 4$

b when $x = \frac{1}{20}$, $(4 + x)^{\frac{1}{2}} \approx 2 + \frac{1}{4}(\frac{1}{20}) - \frac{1}{64}(\frac{1}{20})^2$

$= 2.012\,460\,938$

$(4 + \frac{1}{20})^{\frac{1}{2}} = \sqrt{\frac{81}{20}} = \sqrt{\frac{81\times5}{100}} = \frac{9}{10}\sqrt{5}$

$\therefore\ \sqrt{5} \approx \frac{10}{9} \times 2.012\,460\,938 = 2.236\,067\,71$ (9sf)

c $\sqrt{5} = 2.236\,067\,977\ldots$

$\therefore$ estimate is accurate to 7 significant figures

12 **a** $= 1 + (-\frac{1}{2})(2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(2x)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3\times2}(2x)^3 + \ldots$

$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \ldots$

b $\frac{2-5x}{\sqrt{1+2x}} = (2 - 5x)(1 + 2x)^{-\frac{1}{2}} = (2 - 5x)(1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \ldots)$

$= 2 - 2x + 3x^2 - 5x^3 - 5x + 5x^2 - \frac{15}{2}x^3 + \ldots$

$= 2 - 7x + 8x^2 - \frac{25}{2}x^3 + \ldots$

$\therefore$ for small x, $\frac{2-5x}{\sqrt{1+2x}} = 2 - 7x + 8x^2 - \frac{25}{2}x^3$

c $2 - 5x = \sqrt{3} \times \sqrt{1+2x} = \sqrt{3+6x}$

$(2 - 5x)^2 = 3 + 6x$

$4 - 20x + 25x^2 = 3 + 6x$

$25x^2 - 26x + 1 = 0$

$(25x - 1)(x - 1) = 0$

$x = \frac{1}{25}, 1$

d let $x = \frac{1}{25}$

$\sqrt{3} \approx 2 - 7(\frac{1}{25}) + 8(\frac{1}{25})^2 - \frac{25}{2}(\frac{1}{25})^3$

$= 1.732$

13 **a** $= 1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3\times2}x^3 + \ldots$

$= 1 - x + x^2 - x^3 + \ldots$

b $= 1 - bx + b^2x^2 - b^3x^3 + \ldots$

c $\frac{1+ax}{1+bx} = (1 + ax)(1 + bx)^{-1} = (1 + ax)(1 - bx + b^2x^2 - b^3x^3 + \ldots)$

$= 1 - bx + b^2x^2 - b^3x^3 + ax - abx^2 + ab^2x^3 + \ldots$

$= 1 + (a - b)x + (b^2 - ab)x^2 + (ab^2 - b^3)x^3 + \ldots$

$\therefore \quad a - b = -4 \quad (1)$

and $\quad b^2 - ab = 12 \quad (2)$

(1) $\Rightarrow \quad a = b - 4$

sub. (2) $\quad b^2 - b(b - 4) = 12$

$4b = 12$

$b = 3,\ a = -1$

d $= ab^2 - b^3 = -9 - 27 = -36$